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Hesitant Fuzzy Ideals in Semigroups with Two Frontiers

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Abstract—*We introduce the notions of (ε,δ)-hesitant fuzzy ideal, (ε,δ)-hesitant fuzzy bi-ideal and (ε,δ)-hesitant fuzzy interior ideal on semigroup and characterizations of a hesitant fuzzy ideal (bi-ideal, interior ideal) with two frontiers are considered. We provide an example to show that (ε,δ)-hesitant fuzzy ideal on S need not be a δ-hesitant fuzzy ideal on S. We provide conditions for a (ε,δ)-hesitant fuzzy ideal on S to be a δ-hesitant fuzzy ideal on S. Moreover, we show that the hesitant intersection of two hesitant fuzzy left (right, bi-, interior) ideal with two frontiers is a hesitant fuzzy left (right, bi-, interior) ideal respectively with two frontiers.*

Keywords*: (ε,δ)-hesitant fuzzy semigroup, (ε,δ)-hesitant fuzzy ideal.*

AMS Mathematics Subject classification (2010)*: 20M12, 20M99, 08A72*

1. INTRODUCTION

Given a set, S a fuzzy subset of (or a fuzzy set in S) is described as an arbitrary mapping f: S → [0, 1], where [0,1] is the usual interval of real numbers. The concept of fuzzy set was introduced by Zadeh. The recent generalization of the fuzzy set, Torra[7] initiated the hesitant fuzzy set (briefly HFS) which provides a more specific representation of peoples uncertainty in daily life. HFSs have attracted the attention of many researchers in a short period of time because hesitant situations are very common in different real world problems. HFS theory has been applied to different algebraic structures. The structure of semigroups containing hesitant fuzzy ideals was studied by Jun et al.[3]. Abbasi et al.[1, 2] applied the notion of HFSs to po-semigroups.

Jun et al.[4] introduce the notion of hesitant fuzzy semigroups with frontier and investigate several properties. Later on, many researchers used the idea of hesitant fuzzy semigroups with frontier and gave several results in different branches of algebra. In[6], Aakif et al. applied the concept of hesitant fuzzy semigroups with frontier to the theory of hesitant fuzzy ideals with frontier. Recently, in[5], Jun et al. introduce the notion of hesitant fuzzy semigroups with two frontiers and investigate several properties. In this paper we apply the notion of hesitant fuzzy semigroups with two frontiers to introduce the notion of (ε,δ)--hesitant fuzzy ideals, (ε,δ)--hesitant fuzzy bi-ideals, and (ε,δ)--hesitant fuzzy interior ideals and investigate some related properties.

2. PRELIMINARIES

In this section we discuss some of the basic definitions regarding hesitant fuzzy sets and some of operations and notions on hesitant fuzzy sets are discussed.

For any two subsets X and Y of a semigroup S, the multiplication of X and Y is defined as follows:

$$XY = \{xy \in S \mid x \in X \text{ and } y \in Y\}$$

We now discuss the basic notions on hesitant fuzzy sets. Let S be a reference set, a Hesitant fuzzy set on S is a function H that returns a subset of values in [0,1]:

$$H : S \to P([0,1])$$

where $P([0,1])$ denotes the set of all subsets of [0,1] and $P^*([0,1]) = P([0,1]) \setminus \phi$.

Let H be any hesitant fuzzy set on S and $a,b,c \in$ S, and ε, μ, δ are non-empty subsets of [0, 1]. we use the following notations

$H_a = H(a)$, $H_a^b = H(a) \cap H(b)$, $H_a^b[c] = H(a) \cap H(b) \cap H(c)$, $[H_a](\varepsilon) = H(a) \cap \varepsilon$, $H_a^b(\varepsilon) = H(a) \cap H(b) \cap \in$, $[H_A]_a^b = H_A(a) \cap H_A(b)$, $\sum\{\in,\delta\} = \in \cup \delta$, $\sum\{\in,\delta\}(\mu) = (\in \cup \delta) \cap \mu$, $[H_A]_a = H_A(a)$. It is clear that $H_a^b = H_b^a$ and $H_a = H_b \Leftrightarrow H_a \subseteq H_b$, $H_b \subseteq H_a$ for all $a,b \in S$.

For any two hesitant fuzzy sets H and G on S, let $P([0,1])$ denotes the set of all subsets of [0,1], the hesitant union $H \cup G$ of H and G is defined to be hesitant fuzzy set on S as follows:

$$H \cup G : S \to \mathsf{P}([0,1]), a \mapsto H_a \cup G_a$$

and the hesitant intersection $H \cap G$ of H and G is defined to be hesitant fuzzy set on S as follows:

$$H \cap G : S \to \mathsf{P}([0,1]), a \mapsto H_a \cap G_a.$$

Definition 2.1 [3] *A hesitant fuzzy set H on S is called a hesitant fuzzy semigroup on S if it satisfies :*

$$(\forall\, x, y \in S)(H_x^y \subseteq H_{xy}).$$

Definition 2.2 [4] Let $\varepsilon \in P^*([0,1])$. A hesitant fuzzy set H on S satisfies $H_a^b(\varepsilon) \subseteq H_{ab}$ for all $a, b \in S$, then we say that ε is a frontier of H and H is a hesitant fuzzy semigroup with a frontier ε (briefly ε-hesitant fuzzy semigroup) on S.

Obviously, every hesitant fuzzy semigroup is an ε-hesitant fuzzy semigroup for all $\varepsilon \in P^*([0,1])$. Also, if $\varepsilon \in P([0,1])$ such that $H_a \subseteq \varepsilon$ for all $a \in S$, then every ε-hesitant fuzzy semigroup H on S is a hesitant fuzzy semigroup on S.

3. *MAIN RESULTS*

Definition 3.1 [5] *Let $\varepsilon,\ \delta \in P([0,1])$ with $\varepsilon \neq [0,1]$ and $\delta \neq \emptyset$. A hesitant fuzzy set H on S satisfies $H_a^b(\delta) \subseteq \sum\{H_{ab}, \in\}$ for all $a, b \in S$, then we say that* $\varepsilon,\ \delta$ are frontiers *of H and H is a hesitant fuzzy semigroup with two frontiers ε and δ (briefly (ε, δ)-hesitant fuzzy semigroup) on S.*

Definition 3.2 [6] *A hesitant fuzzy set H on S is called a ε-hesitant fuzzy left (resp., right) ideal on S if it satisfies :*

$$H_b(\varepsilon) \subseteq H_{ab}\ (resp.,\ H_a(\varepsilon) \subseteq H_{ab})$$
$$(\forall a, b \in S\ \&\ \varepsilon \in \mathsf{P}^*([0,1])).$$

If a hesitant fuzzy set H on S is both a ε-hesitant fuzzy left ideal and a ε- hesitant fuzzy right ideal on S, we say that H is a hesitant ε-fuzzy two-sided ideal on S.

Definition 3.3 A hesitant fuzzy set H on S is called a (ε, δ)-hesitant fuzzy left (resp., right) ideal on S if it satisfies :

$$H_b(\delta) \subseteq \sum\{H_{ab}, \in\}\ (resp.,\ H_a(\delta) \subseteq \sum\{H_{ab}, \in\})$$
$$\forall\, a, b, c \in S\ \&\ \ \varepsilon \neq [0,1]\ ,\ \delta \neq \emptyset\ .$$

If a hesitant fuzzy set H on S is both a (ε, δ)-hesitant fuzzy left ideal and a (ε, δ) - hesitant fuzzy right ideal on S, we say that H is a hesitant (ε, δ)-hesitant fuzzy two-sided ideal on S.

For any $\varepsilon,\ \delta \in P([0,1])$ *with* $\varepsilon \neq [0,1]$ *and* $\delta \neq \emptyset$, every δ-hesitant fuzzy ideal on S is a (ε, δ)-hesitant fuzzy ideal on S but converse need not be true in general which is shown by the following example.

Example 3.1 Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table :

.	**a**	**b**	**c**	**d**
a	b	b	a	b
b	b	b	b	b
c	a	b	c	b
d	b	b	d	b

Clearly S is a semigroup.

Define a hesitant fuzzy subset H of S such that

$H_a = \{0.1,0.2,0.3\}, H_b = \{0.4,0.5\},\ H_c =\ H_d = \{0.2,03,0.6\}$. Let $\varepsilon = \{0.1,0.2,0.3\}$ and $\delta = \{0.1,0.3\}$. Then H is (ε, δ)-hesitant fuzzy ideal on S. But it is not a δ-hesitant fuzzy ideal on S since

$$H_a \cap \delta = \{0.1,0.3\} \not\subset H_{aa} = \{0.4,0.5\}.$$

Then (ε, δ)-hesitant fuzzy ideal on S need not be a δ-hesitant fuzzy ideal on S.

We provide conditions for a (ε, δ)-hesitant fuzzy ideal on S to be a δ-hesitant fuzzy ideal on S .

Theorem 3.1. For any $\varepsilon,\ \delta \in P([0,1])$ *with* $\varepsilon \neq [0,1]$ *and* $\delta \neq \emptyset$, if ε and δ are disjoint, then every (ε, δ)-hesitant fuzzy left (right) ideal on S to be a δ-hesitant fuzzy left (right) ideal on S.

Proof. Let H be a (ε,δ)-hesitant fuzzy left ideal on S. Then $H_b(\delta)\subseteq\sum\{H_{ab},\in\}$ for all a, b ∈ S. If k ∈ $H_b(\delta)$ then k ∈ $\sum\{H_{ab},\in\}$ and k ∈ δ. Since ε and δ are disjoint, it follows that k $\notin \varepsilon$ and that k ∈ H_{ab}. Hence $H_b(\delta)\subseteq H_{ab}$ for all a, b ∈ S. Therefore H is δ-hesitant fuzzy left ideal on S.

Similarly we can prove for another case.

Theorem 3.2. *The hesitant intersection of two (ε,δ)-hesitant fuzzy right (resp., left) ideal on S is an (ε,δ)-hesitant fuzzy right (resp., left) ideal on S.*

Proof. Let H and G be two (ε,δ)-hesitant fuzzy right ideal on S. For any $a,b\in S$, we have $\sum\{H_{ab},\in\}\supseteq H_a(\delta)$ and $\sum\{G_{ab},\in\}\supseteq G_a(\delta)$. We will prove that $\sum\{(H\cap G)_{ab},\in\}\supseteq (H\cap G)_a(\delta)$

$$\sum\{(H\cap G)_{ab},\in\} = (H_{ab}\cap G_{ab})\cup\in$$

$$=(H_{ab}\cup\in)\cap(G_{ab}\cup\in)$$

$$=\sum\{H_{ab},\in\}\cap\sum\{G_{ab},\in\}$$

$$\supseteq (H_a(\delta))\cap(G_a(\delta))$$
$$=(H_a\cap G_a)\cap\delta$$
$$=(H\cap G)_a\cap\delta$$
$$=(H\cap G)_a(\delta).$$

Hence $H\cap G$ is an (ε,δ)-hesitant fuzzy right ideal on S. Similarly, we can prove the other case.

Theorem 3.3 Let $\varepsilon,\ \delta\in P([0,1])$ *with* $\varepsilon\neq[0,1]$ *and* $\delta\neq\emptyset$, *such that* $\delta\subseteq H_x\ \forall$ x S. Then H *be an* (ε,δ)-*hesitant fuzzy ideal on* S *if and only forall* $a,b\in S$;

$$\sum\{H_a,\in\}\supseteq\delta,\sum\{H_b,\in\}\supseteq\delta\Rightarrow\sum\{H_{ab},\in\}\supseteq\delta.$$

Proof. Assume that H is an (ε,δ)-hesitant fuzzy ideal on S. For any $a,b\in S,$ we have $\sum\{H_{ab},\in\}\supseteq H_a(\delta)=\delta$. Also, $\sum\{H_{ab},\in\}\supseteq H_b(\delta)=\delta.$

Conversely, suppose $\forall\, a,b\in S$ such that $\sum\{H_a,\in\}\supseteq\delta,\sum\{H_b,\in\}\supseteq\delta\Rightarrow\sum\{H_{ab},\in\}\supseteq\delta$. It follows that $\sum\{H_{ab},\in\}\supseteq\delta\supseteq H_a(\delta)$ and $\sum\{H_{ab},\in\}\supseteq\delta\supseteq H_b(\delta)$. Hence H be an (ε,δ)-hesitant fuzzy ideal on S.

Theorem 3.4 Let $\varepsilon,\ \delta\in P([0,1])$ *with* $\varepsilon\neq[0,1]$ *and* $\delta\neq\emptyset$. *If* H *is an* (ε,δ)-*hesitant fuzzy ideal on* S *such that* $H_a\subseteq H_b$. *Then the set* $S_a=\{b\in S\mid\sum\{H_b,\in\}\supseteq H_a(\delta)\}$ *is an ideal of* S.

Proof. Since $a\in S_a\ \forall\, a\in S, S_a\neq\emptyset.$ Let b be any element of S_a and c be any element of S. Then $\sum\{H_b,\in\}\supseteq H_a(\delta)$. Since H is an (ε,δ)-hesitant fuzzy ideal on S and $H_a\subseteq H_b$, we have $\sum\{H_{bc},\in\}\supseteq H_b(\delta)\supseteq H_a(\delta)$ implies that $bc\in S_a$. Similarly we have, $cb\in S_a$. Hence, S_a is an ideal of S for all $a\in S.$

Definition 3.4 [3] *A hesitant fuzzy subsemigroup* H *on* S *is called a hesitant fuzzy bi-ideal on* S *if it satisfies:*

$$(H_{xyz}\supseteq H_x^z)\ (\forall\, x,y,z\in S).$$

Definition 3.5 *A hesitant fuzzy set* H *on* S *is called a* (ε,δ)-*hesitant fuzzy bi-ideal on* S *if* $\forall a,b,x\in S$ & $\varepsilon\neq[0,1]$, $\delta\neq\emptyset$ *it satisfies :*

(i) $H_a^b(\delta)\subseteq\sum\{H_{ab},\in\}$

(ii) $H_a^b(\delta)\subseteq\sum\{H_{axb},\in\}$

Theorem 3.5 *The hesitant intersection of two* (ε,δ)-*hesitant fuzzy bi-ideals on* S *is an* (ε,δ)-*hesitant fuzzy bi-ideal on* S.

Proof. Let H and G be two (ε,δ)-hesitant fuzzy bi-ideals on S. For any $a,b,x\in S$, we have

$$\sum\{(H\cap G)_{ab},\in\} = (H_{ab}\cap G_{ab})\cup\in$$

$= (H_{ab}\cup\in)\cap(G_{ab}\cup\in)$

$\supseteq (H_a\cap H_b\cap(\delta))\cap(G_a\cap G_b\cap(\delta))$

$= (H_a\cap G_a)\cap(H_b\cap G_b)\cap\delta$

$= (H\cap G)_a\cap(H\cap G)_b\cap\delta$
$= (H\cap G)_a^b(\delta).$

and

$\sum\{(H\cap G)_{axb},\in\} = (H_{axb}\cap G_{axb})\cup\in$

$= (H_{axb}\cup\in)\cap(G_{axb}\cup\in)$

$\supseteq (H_a\cap H_b\cap(\delta))\cap(G_a\cap G_b\cap(\delta))$

$= (H_a\cap G_a)\cap(H_b\cap G_b)\cap\delta$

$= (H\cap G)_a\cap(H\cap G)_b\cap\delta$
$= (H\cap G)_a^b(\delta).$

Hence $H\cap G$ is an (ε,δ) -hesitant fuzzy bi-ideal on S.

Definition 3.6 [3] *A hesitant fuzzy subsemigroup H on S is called a hesitant fuzzy interior idea on S if it satisfies:*

$$(H_{xyz}\supseteq H_y)\ (\forall\ x,y,z\in S).$$

Definition 3.7 *A hesitant fuzzy set H on S is called a (ε,δ) -hesitant fuzzy interior ideal on S if $\forall a,b,x\in S$ & $\varepsilon\neq[0,1]$, $\delta\neq\emptyset$ it satisfies:*

(i) $H_a^b(\delta)\subseteq\sum\{H_{ab},\in\}$

(ii) $H_a(\delta)\subseteq\sum\{H_{xay},\in\}$

Theorem 3.6 *The hesitant intersection of two (ε,δ) -hesitant fuzzy interior ideals on S is an (ε,δ) -hesitant fuzzy interior ideal on S.*

Proof. Let H and G be two (ε,δ) -hesitant fuzzy interior ideals on S. For any $a,b,x\in S$, we have

$$\sum\{(H\cap G)_{ab},\in\} = (H_{ab}\cap G_{ab})\cup\in$$

$= (H_{ab}\cup\in)\cap(G_{ab}\cup\in)$

$\supseteq (H_a\cap H_b\cap(\delta))\cap(G_a\cap G_b\cap(\delta))$

$= (H_a\cap G_a)\cap(H_b\cap G_b)\cap\delta$
$= (H\cap G)_a\cap(H\cap G)_b\cap\delta$
$= (H\cap G)_a^b(\delta).$

and

$\sum\{(H\cap G)_{axb},\in\} = (H_{axb}\cap G_{axb})\cup\in$

$= (H_{axb}\cup\in)\cap(G_{axb}\cup\in)$

$\supseteq (H_x\cap(\delta))\cap(G_x\cap(\delta))$
$= (H_x\cap G_x)\cap\delta$
$= (H\cap G)_x\cap\delta$

Hence $H\cap G$ is an (ε,δ) -hesitant fuzzy interior ideal on S.

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